

On Contact Graphs with Cubes and Proportional Boxes

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Abstract. We study two variants of the problem of contact representation of planar graphs with axis-aligned boxes. In a *cube-contact representation* we realize each vertex with a cube, while in a *proportional box-contact representation* each vertex is an axis-aligned box with a prespecified volume. We present algorithms for constructing cube-contact representation and proportional box-contact representation for several classes of planar graphs.

1 Introduction

We study *contact representations* of planar graphs in 3D, where vertices are represented by interior-disjoint axis-aligned boxes and edges are represented by shared boundaries between the corresponding boxes. A contact representation of a planar graph G is *proper* if for each edge (u, v) of G , the boxes for u and v have a shared boundary with non-zero area. Such a contact between two boxes is also called a *proper contact*. *Cubes* are axis-aligned boxes where all sides have the same length. A contact representation of a planar graph with boxes is called a *cube-contact representation* when all the boxes are cubes. In a weighted variant of the problem a *proportional box-contact representation* is one where each vertex v is represented with a box of volume $w(v)$, for any function $w : V \rightarrow \mathbb{R}^+$, assigning weights to the vertices V . Note that this “value-by-volume” representation is a natural generalization of the “value-by-area” cartograms in 2D.

Related Work: The history of representing planar graphs as contact graphs dates back at least to Koebe’s 1930 theorem [13] for representing planar graphs by touching disks in 2D. Proper contact representation with rectangles in 2D is the well-known *rectangular dual* problem, for which several characterizations exist [14, 18]. Representations with other axis-aligned and non-axis-aligned polygons [7, 11, 19] have been studied. Related graph-theoretic, combinatorial and geometric problems continue to be of interest [6, 8, 12]. The weighted variant of the problem has been considered in the context of rectangular, rectilinear, and unrestricted cartograms [4, 9, 15].

Contact representations have been also considered in 3D. Thomassen [17] shows that any planar graph has a proper contact representation with touching boxes, while Felsner and Francis [10] find a (not necessarily proper) contact representation of any planar graph with touching cubes. Recently, Bremner *et al.* [5] asked whether any planar graph can be represented by proper contacts of cubes. They answered the question positively for the case of partial planar 3-trees and some planar grids, but the problem remains open for general planar graphs. The weighted variant of the problem in 3D is

much less studied, although recently Alam *et al.* [1] have presented algorithms for proportional representation of several classes of graphs (e.g., outerplanar, planar bipartite, planar, complete), using 3D L-shapes.

Our Contribution: Here we expand the class of planar graph representable by proper contact of cubes. We also show that several classes of planar graphs admit proportional box-contact representations. Specifically, we show how to compute a proportional box-contact representation for plane 3-trees, while a cube-contact representation for the same graph class follows from [5]. We also show how to compute a proportional box-contact representation and a cube-contact representation for *nested maximal outerplanar graphs*, which are defined as follows. A *nested outerplanar graph* is either an outerplanar graph or a planar graph G where each component induced by the internal vertices is another nested outerplanar graph with exactly three neighbors in the outface of G . A *nested maximal outerplanar graph* is a subclass of nested outerplanar graphs that is either a maximal outerplanar graph or a maximal planar graph in which the vertices on the outface induce a maximal outerplanar graph and each component induced by internal vertices is another nested maximal outerplanar graph.

2 Preliminaries

A *3-tree* is either a 3-cycle or a graph G with a vertex v of degree three in G such that $G - v$ is a 3-tree and the neighbors of v form a triangle. If G is planar, then it is called a *planar 3-tree*. A *plane 3-tree* is a planar 3-tree along with a fixed planar embedding. Starting with a 3-cycle, any planar 3-tree can be formed by recursively inserting a vertex inside a face and adding an edge between the newly added vertex and each of the three vertices on the face [3, 16]. Using this simple construction, we can create in linear time a *representative tree* for G [16], which is an ordered rooted ternary tree T_G spanning all the internal vertices of G . The root of T_G is the first vertex we have to insert into the face of the three outer vertices. Adding a new vertex v in G will introduce three new faces belonging to v . The first vertex w we add in each of these faces will be a child of v in T_G . The correct order of T_G can be obtained by adding new vertices according to the counterclockwise order of the introduced faces.

An *outerplanar graph* is one that has a planar embedding with all vertices on the same face (outface). An outerplanar graph is *maximal* if no edge can be added without violating its outerplanarity. Thus in a maximal outerplanar graph all the faces except for the outface are triangles. For $k > 1$, a k -outerplanar graph G is an embedded graph such that deleting the outer-vertices from G yields a graph where each component is at most a $(k - 1)$ -outerplanar graph; a 1-outerplanar graph is just an outerplanar graph. Note that any planar graph is a k -outerplanar graph for some integer $k > 0$.

Let G be a planar graph. We define the *pieces* of G as follows. If G is outerplanar, it has only one piece, the graph itself. Otherwise, let G_1, G_2, \dots, G_l be the components of the graph obtained by deleting the outer vertices (and their incident edges) from G . Then the pieces of G are all the pieces of G_i for each $i \in \{1, 2, \dots, l\}$, as well as the subgraph of G induced by the outer-vertices of G . Note that each piece of G is an outerplanar graph. Since G is an embedded graph, for each piece P of G , we can define the *interior* of P as the region bounded by the outer cycle of P . Then we can

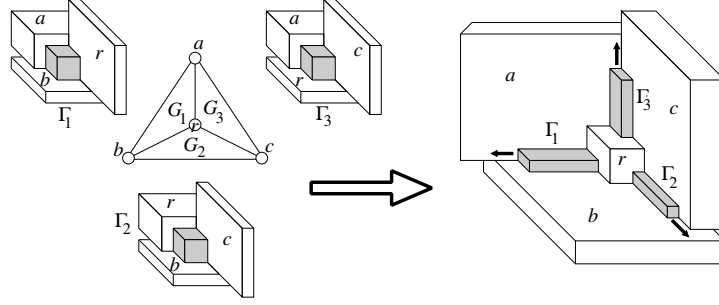


Fig. 1. Illustration for the proof of Theorem 1.

define a rooted tree \mathcal{T} where the pieces of G are the vertices of \mathcal{T} and the parent-child relationship in \mathcal{T} is determined as follows: for each piece P of G , its children are all the pieces of G that are in the interior of P but not in the interior of any other pieces of G . A piece of G has *level* l if it is on the l -th level of \mathcal{T} . All the vertices of a piece at level l are also l -level vertices. A planar graph is a *nested outerplanar graph* if each of its pieces at level $l > 0$ has exactly three vertices of level $(l - 1)$ as a neighbor of some of its vertices. On the other hand a *nested maximal outerplanar graph* is a maximal planar graph where all the pieces are maximal outerplanar graphs.

3 Representations for Planar 3-trees

Here we prove that planar 3-trees have proportional box-representations in two different ways. The first one is a more intuitive proof; the second one includes a direct computation of the coordinates for the representation.

Theorem 1. *Let $G = (V, E)$ be a plane 3-tree with a weight function w . Then a proportional box-contact representation of G can be computed in linear time.*

First Proof: Let a, b, c be the outer vertices of G . We construct a representation Γ for G where b occupies the bottom side of Γ , a occupies the back of $\Gamma - \{b\}$ and c occupies the right side of $\Gamma - \{a, b\}$; see Fig. 1. Here for a set of vertices S , $\Gamma - S$ denotes the representation obtained from Γ by deleting the boxes representing the vertices in S . The claim is trivial when G is a triangle, so assume that G has at least one internal vertex. Let r be the root of the representation tree T_G of G . Then r is adjacent to a, b and c and thus defines three regions G_1, G_2 and G_3 inside the triangles $\Delta_1 = abr$, $\Delta_2 = bcr$ and $\Delta_3 = car$, respectively (including the vertices of these triangles). By induction hypothesis $G_i, i = 1, 2, 3$ has a proportional box-contact representation Γ_i where the boxes for the three vertices in Δ_i occupy the bottom, back and right sides of Γ_i . Define $\Gamma'_i = \Gamma_i - \Delta_i$. We now construct the desired representation for G . First take a box for r with volume $w(r)$ and place it in a corner created by the intersection of three pairwise-touching boxes; see Fig. 1. For each $\Delta_i, i = 1, 2, 3$, there is a corner p_i formed by the intersection of the three boxes for Δ_i . We now place Γ'_i (after possible scaling)

in the corner p_i so that it touches the boxes for the vertices in Δ_i by three planes. Note that this is always possible since we can choose the surface areas for a , b and c to be arbitrarily large and still realize their corresponding weights by appropriately changing the thickness in the third dimension. This construction requires only linear time, by keeping the scaling factor for each region in the representative tree T_G at the vertex representing that region. Then the exact coordinates can be computed with a top-down traversal of T_G . \square

Second Proof: Assume (after possible factoring) that for each vertex v of G , the weight $w(v)$ is at least 1. Let T_G be the representative tree of G . For any vertex v of T_G , we denote by U_v , the set of the descendants of v in T_G including v . The *predecessors* of v are the neighbors of v in G that are not in U_v . Clearly each vertex of T_G has exactly three predecessors. We now define a parameter $W(v)$ for each vertex v of T_G . Let v_1 , v_2 and v_3 be the three children of v in T_G (where zero or more of these three children may be empty). Then $W(v)$ is defined as $\prod_{i=1}^3 [W(v_i) + \sqrt[3]{w(v)}]$, where $U(v_i)$ is taken as zero when v_i is empty. We can compute the value of $W(v)$ for each vertex v of T_G by a linear-time bottom-up traversal of T_G . Once we have computed these values, we proceed on constructing the box-contact representation as follows.

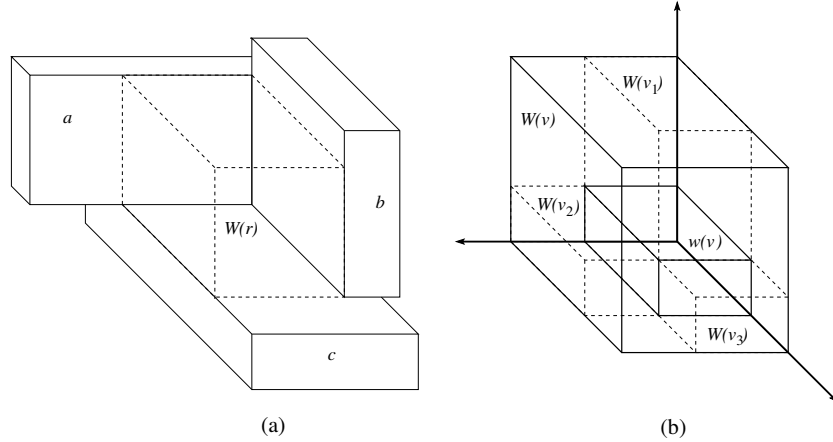


Fig. 2. Illustration for the second proof of Theorem 1.

Let a , b , c be the three outer vertices of G in the clockwise order and let r be the root of T_G . We start by computing three boxes for a , b and c with the correct volume as illustrated in Fig. 2(a), so that the volume of the dotted box R is $W(r)$. We will now construct a box representation of U_r inside R so that all the vertices in U_r adjacent to an outer vertex is represented by a box with a face co-planar on the face of R adjacent to box representing that outer vertex. We do this recursively by a top-down computation on T_G . Let v be a vertex of T_G with the three predecessors u_1 , u_2 and u_3 . Let $D(v)$ be a box with volume $W(v)$ and let t_1 , t_2 , t_3 be three faces of it with a common point. While traversing v , we compute a proportional box-contact representation of U_v inside $D(v)$

where the vertices in U_v adjacent to u_i for some $i \in \{1, 2, 3\}$ is represented by a box with a face co-planar with t_i . Let v_1, v_2 and v_3 are the three children of v in T_G (where zero or more of these children may be empty). Also assume that x_1, x_2, x_3 are the length, width and height of $D(v)$, respectively and p is the common point of t_1, t_2 and t_3 . Then first compute a box $R(v)$ of volume $w(v)$ for v with a corner at p where x'_1, x'_2 and x'_3 are the length, width and height of $R(v)$, such that $x'_i = \frac{\sqrt[3]{w(v)}}{W(v_1) + \sqrt[3]{w(v)}}$. These choices of x'_i 's also creates three boxes $D(v_i)$ with volume at least $W(v_i)$, $i \in \{1, 2, 3\}$, as illustrated in Fig. 2(b). Finally we recursively compute the box representations for U_{v_i} inside $D(v_i)$ for $i \in \{1, 2, 3\}$ to complete the construction. \square

Theorem 2. [5] *Let G be a plane 3-tree. Then a cube-contact representation of G can be computed in linear time.*

The proof of this claim also relies on the recursive decomposition of planar 3-trees.

4 Cube-Contacts for Nested Maximal Outerplanar Graphs

We prove the following main theorem in this section:

Theorem 3. *Any nested maximal outerplanar graph has a proper contact representation with cubes.*

We prove Theorem 3 by construction, starting with a representation for each piece of G , and combining the pieces to complete the representation for G .

Let G be a nested maximal outerplanar graph. We first augment the graph G by adding three mutually adjacent dummy vertices $\{A, B, C\}$ on the outerface and then triangulating the graph by adding dummy edges from these three vertices to the outer vertices of G such that the graph remains planar; see Fig. 4(a). Call this graph the *extended graph* of G . For consistency, let the three dummy vertices have level 0. The observation below follows from the definition of nested maximal outerplanar graphs.

Observation 1 *Let G be a nested-maximal planar graph and let G' be the extended graph of G . Then for each piece P of G at level l , there is a triangle of $(l - 1)$ -level vertices adjacent to the vertices of P and no other k -level vertices with $k < l$ are adjacent to any vertex of P .*

Given this observation, we use the following strategy to obtain a contact representation of G with cubes. For each piece P of G at level l , let A, B and C be the three $(l - 1)$ -level vertices adjacent to P 's vertices. Let P' be the subgraph of G induced vertices of P as well as A, B and C ; call P' the *extended piece* of G for P . We obtain a contact representation of P' with cubes and delete the three cubes for A, B and C to obtain the contact representation of P with cubes. Finally, we combine the representations for the pieces to complete the desired representation of G .

Before we give more details on this algorithm, we have the following lemma, that we use in this section. Furthermore this result is also interesting by itself, since for any outerplanar graph O , where each face has at least one outer edge, Lemma 2 gives a contact representation of O on the plane with squares such that the outer boundary of the representation is a rectangle.

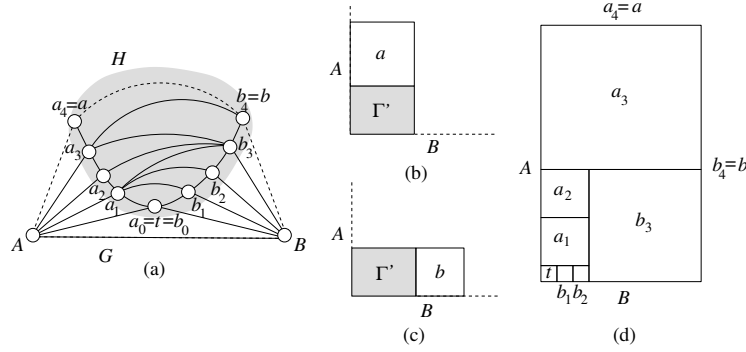


Fig. 3. Illustration for the proof of Lemma 2.

Lemma 2. *Let G be planar graph with outerface $ABba$ and at least one internal vertex, such that $G - \{A, B\}$ is a maximal outerplanar graph. If there is no chord between any two neighbors of A and no chord between any two neighbors of B , then G has a contact representation Γ in 2D where each inner vertex is represented by a square, the union of these squares forms a rectangle, and the four sides of these rectangles represent A, B, b and a , respectively.*

Proof: We prove this lemma by induction on the number of vertices in G . Denote the maximal outerplanar graph $H = G - \{A, B\}$; see Fig. 3(a). If G contains only one internal vertex v , then we compute Γ by representing v by a square $R(v)$ of arbitrary size and representing A, B, b and a by the left, bottom, right and top sides of $R(v)$.

We thus assume that G has at least two internal vertices. Let u be the unique common neighbor of $\{a, b\}$ in H . If u is a neighbor of A , then $H - \{a\}$ is a maximal outerplanar graph. By induction hypothesis, $G - \{a\}$ has a contact representation Γ' where each internal vertex of $G - \{a\}$ is represented by a square and the left, bottom, right and top sides of Γ' represent A, B, b and u . Then we compute Γ from Γ' by adding a square $R(u)$ to represent u such that $R(u)$ spans the entire width of Γ' and is placed on top of Γ' ; see Fig. 3(b). A similar construction can be used if u is a neighbor of B ; see Fig. 3(c). We thus compute a contact representation for G ; see Fig. 3(d). \square

4.1 Cube-Contact Representation for Extended Pieces

Lemma 3. *Let P be a piece of G at level l and P' be the extended piece for P with $(l - 1)$ -level vertices A, B, C . Then P' has a cube-contact representation.*

Proof: Let r be a common neighbor of B and C ; s a common neighbor of A and C ; t a common neighbor of A and B . It is easy to find a contact representation of P' if r, s and t are the only vertices of P , so let P have at least four vertices. The outer cycle of P can be partitioned into three paths: P_a is the path from s to t , P_b is the path from r to t and P_c is the path from r to s . Note that all vertices on the path P_a (P_b, P_c) are adjacent to A (B, C). A chord (u, v) is a *short chord* if it is between two vertices on the

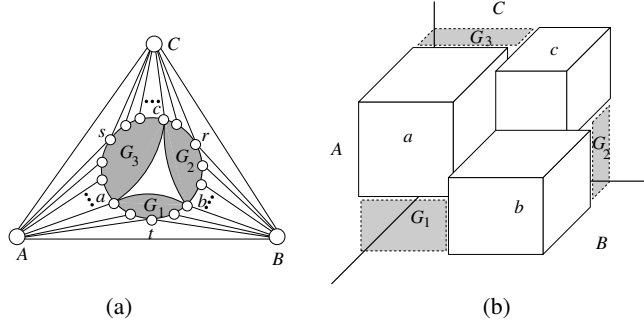


Fig. 4. Illustration for **Case A1** in the proof of Lemma 3.

same path from the set $\{P_a, P_b, P_c\}$. (Note that a chord between two vertices from the set $\{r, s, t\}$ is also a short chord.) We have the following two cases.

Case A: There is no short chord in P . In this case all the chords of P are between two different paths. We consider the following two subcases.

Case A1: There is no chord with one end-point in $\{r, s, t\}$. In this case, due to maximal-planarity there exist three vertices a, b and c , adjacent to A, B , and C , respectively such that (i) ab is the chord between vertices of P_a and P_b farthest away from t , (ii) bc is the chord between vertices of P_b and P_c farthest away from r , and (iii) ac is the chord between vertices of P_a and P_c farthest away from s ; see Fig. 4(a). We can then find three interior-disjoint subgraphs of P' defined by three cycles of P' : G_1 is the one induced by all vertices on or inside $ABba$; G_2 is induced by all vertices on or inside $BCcb$; and G_3 is induced by all vertices on or inside $ACca$. Each of these subgraphs has the common property that if we delete two vertices from the outerface (two vertices from the set $\{A, B, C\}$ in each subgraph), we get an outerplanar graph. From the representation with squares from the proof of Lemma 2, we find a contact representation of $G_i, i = 1, 2, 3$ where each internal vertex of G_i is represented by a cube and the union of all these cubes forms a rectangular box whose four sides realize the outer vertices. We use such a representation to obtain a contact representation of P' with cubes as follows.

We draw pairwise adjacent cubes (of arbitrary size) for A, B, C . We need to place the cubes for all the vertices of P in the a corner defined by three faces of the cubes for A, B, C . Then we place three mutually touching cubes for a, b and c , which touch the walls for A, B and C , respectively; see Fig. 4(b). We also compute a contact representation of the internal vertices for each of the three graphs G_1, G_2 and G_3 with cubes using Lemma 2, so that the outer boundary for each of these representation forms a rectangular pipe. We adjust the sizes of the three cubes for a, b and c in such a way that the three highlighted rectangular pipes precisely fit these three representations (after some possible scaling). Note that this construction works even if one or more of the subgraphs G_1, G_2 and G_3 are empty. This completes the analysis of **Case A1**.

Case A2: There is at least one chord with one end-point in $\{r, s, t\}$. Due to planarity all such chords will have the same end-point in $\{r, s, t\}$. Suppose s is this common end point for these chords; see Fig. 5(a). Let b_1 and b_f be the first and last

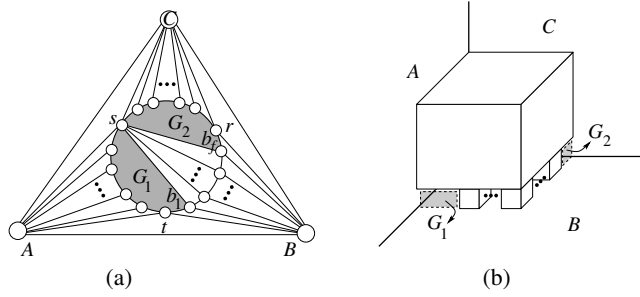


Fig. 5. Illustration for **Case A2** in the proof of Lemma 3.

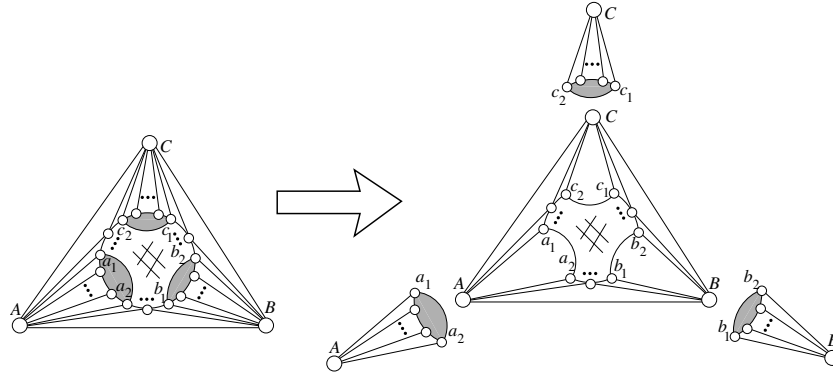


Fig. 6. Removing chords with end-vertices in the same neighborhood.

endpoints in the clockwise order of these chords around s . Then we can find two subgraphs G_1 and G_2 induced by the vertices on or inside two separating cycles ABb_1s and $BCsb_f$. We find contact representations for the internal vertices of these two graphs G_1 and G_2 using Lemma 2 so that the outer-boundaries of these representation form rectangular pipes. We then obtain the desired contact representation for P' , starting with the three mutually touching walls for A , B and C at right angles from each other, placing the cubes for s and b_1, \dots, b_f as illustrated in Fig. 5(b), and fitting the representations for G_1 and G_2 (after some possible scaling) in the highlighted regions.

Case B: there are some shord chords in P . In this case, we find at most four subgraphs from P' as follows. At each path in $\{P_a, P_b, P_c\}$, we find the *outermost chord*, i.e., one that is not contained inside any other chords on the same path. Suppose these chords are a_1a_2 , b_1b_2 and c_1c_2 , on the three paths P_a , P_b , P_c , respectively. Then three of these subgraphs G_a , G_b and G_c are induced by the vertices on or inside the three triangles Aa_1a_2 , Bb_1b_2 and Cc_1c_2 . The fourth subgraph P^* is obtained from P' by deleting all the inner vertices of the three graphs G_a , G_b and G_c ; see Fig. 6.

A cube representation of P^* can be found by the algorithm in **Case A**, as P^* fits the condition that there is no chord between any two neighbors of the same vertex in $\{A, B, C\}$. Note that by moving the cubes in the representation by an arbitrarily small

amount, we can make sure that for each triangle xyz in P^* , the three cubes for x , y and z form a corner surrounded by three mutually touching walls at right angles to each other. Now observe that each of the three graphs G_a , G_b and G_c is a planar 3-tree; thus using the algorithm of either [5] or [10], we can place the internal vertices of these three graphs in their corresponding corners, thereby completing the representation. \square

4.2 Cube-Contact Representation for a Nested Maximal Outerplanar Graph

Proof of Theorem 3: Let G be a nested maximal outerplanar graph. We build the contact representation of G by a top-down traversal of the rooted tree \mathcal{T} of the pieces of G . We start by creating a corner surrounded by three mutually touching walls at right angle to each other. Then whenever we traverse any vertex of \mathcal{T} , we realize the corresponding piece P at level l by obtaining a representation using Lemma 3 and placing this in the corner created by the three already-placed cubes for the three $(l-1)$ -level vertices adjacent to P (after possible scaling). \square

5 Proportional Box-Contacts for Nested Outerplanar Graphs

In this section we prove the following main theorem.

Theorem 4. *Let $G = (V, E)$ be a nested outerplanar graph and let $w : V \rightarrow \mathbb{R}^+$ be a weight function defining weights for the vertices of G . Then G has a proportional contact representation with axis-aligned boxes with respect to w .*

We construct a proportional representation for G using a similar strategy as in the previous section: we traverse the construction tree \mathcal{T} of G and deal with each piece of G separately. Each piece P of G is an outerplanar graph and hence one can easily construct a proportional box-contact representation for P as follows. Any outerplanar graph P has a contact representation with rectangles in the plane. In fact in [2], it was shown that P has a contact representation with rectangles on the plane where the rectangles realize prespecified weights by their areas. Thus by giving unit heights to all rectangles we can obtain a proportional box-contact representation of P for any given weight function. However if we construct proportional box-contact representation for each piece of G in this way, it is not clear that we can combine them all to find a proportional contact representation of the whole graph G . Instead, we use this construction idea in Lemmas 4 and 5 to build two different proportional rectangle-contact representations for outerplanar graphs and we use them in the proof of Theorem 4.

Suppose O is an outerplanar graph and Γ is a contact representation of O with rectangles in the plane. We say that a corner of a rectangle in Γ is *exposed* if it is on the outer-boundary of Γ and is not shared with any other rectangles.

Lemma 4. *Let O be a maximal outerplanar graph with a weight function w . Let $1, \dots, n$ be the clockwise order of the vertices around the outer-cycle. Then a proportional rectangle-contact representation Γ of O for w can be computed so that rectangle R_1 for 1 is leftmost in Γ , rectangle R_n for n is bottommost in $\Gamma - R_1$, and the top-right corner for each rectangle is exposed in Γ .*

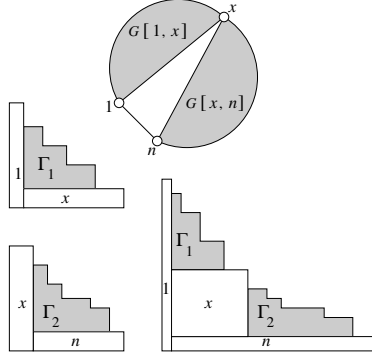


Fig. 7. Illustration for the proof of Lemma 4.

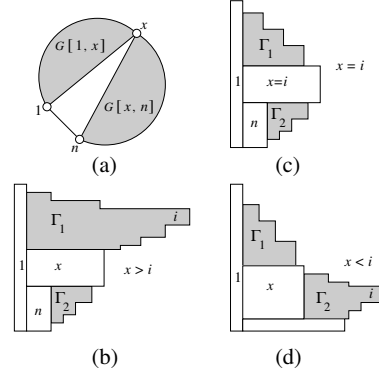


Fig. 8. Illustration for the proof of Lemma 5.

Proof: We give an algorithm that recursively computes Γ . Constructing Γ is easy when G is a single edge $(1, n)$. We thus assume that G has at least 3 vertices. Let x be the (unique) third vertex on the inner face that is adjacent to $(1, n)$. Then graph G can be split into two graphs at vertex x and edge $(1, n)$: $G[1, x]$ consists of the graph induced by all vertices between 1 and x in clockwise order around the outer-cycle; while $G[x, n]$ consists of the graph induced by the vertices between x and n .

Recursively draw $G[1, x]$ and remove the rectangles for 1 and x from it; call the result Γ_1 . Again recursively draw $G[x, n]$ and remove x and n from it; call the result Γ_2 . Now draw a rectangle R_x for x with area $w(x)$. Let l_x and h_x be the width and height of R_x , respectively. Then draw the rectangles R_1 and R_n for 1 and n touching the left and the bottom sides of R_x , respectively with necessary areas. Select the widths and heights of these two rectangles such that the area $l_x(h_1 - h_n - h_x)$ can contain Γ_1 while the area $(w_n - w_x) * h_x$ can contain Γ_2 , where l_j and h_j denote the width and height of R_j , respectively for $j \in \{1, n\}$. Finally place Γ_1 (after possible scaling) touching the right side of R_1 and the top side of R_x and place Γ_2 (after possible scaling) touching the right side of R_x and the top side of R_n to complete the drawing; see Fig. 7. \square

Note that in the layout obtained above the top right corners of the rectangles for vertices $\{1, \dots, n\}$ have increasing x -coordinates and decreasing y -coordinates. Thus we refer to them as **Staircase** layouts and to the algorithm as the **Staircase Algorithm**.

Lemma 5. *Let O be a maximal outerplanar graph with a weight function w . Let $1, \dots, n$ be the clockwise order of the vertices around the outer-cycle. Then a proportional rectangle-contact representation Γ of O for w can be computed so that rectangle R_1 for 1 is leftmost in Γ , rectangle R_n for n is bottommost in $\Gamma - R_1$, and the top-right corners of all rectangles for vertices $\{1, \dots, i\}$ and the bottom-right corners of all rectangles for vertices $\{i, \dots, n\}$ are exposed in Γ .*

Proof: We again compute Γ recursively. Constructing Γ is easy when G is a single edge $(1, n)$. We thus assume that G has at least 3 vertices. Let x be the (unique) third

vertex on the inner face that is adjacent to $(1, n)$. Define the two graphs $G[1, x]$ and $G[x, n]$ as in the proof of Lemma 4; see also Fig. 8(a).

If $x > i$, then recursively draw $G[1, x]$ and remove the rectangles for 1 and x from it; call the result Γ_1 . Draw $G[x, n]$ using the **Staircase Algorithm**. Now draw three mutually touching rectangles R_1, R_x and R_n for 1, x and n , respectively with necessary areas such that the right side of R_1 touches both R_x and R_n and the right side of R_x has greater x -coordinate than the right side of R_n ; see Fig. 8(b). Finally place Γ_1 (after possible scaling) touching the right side of R_1 and the top side of R_x such that the right side of the rectangle for $(x - 1)$ extends past R_x . Also place Γ_2 (after 90° clockwise rotation and possible scaling) touching the bottom side of R_x and the right side of R_n to complete the drawing (the width of R_x can be chosen long enough so that Γ_2 can be contained between the bottom side of R_x and the right side of R_n).

On the other hand if $x = i$ we follow almost the same procedure as in the previous paragraph. However, instead of the drawing of $G[1, x]$ recursively, we compute it by the **Staircase Algorithm** and then delete from it 1 and x to obtain Γ_1 . We compute Γ_2 as in the previous section. We also draw R_1, R_x and R_n in the same way. Then we place Γ_1 (after possible scaling) touching the right side of R_1 and the top side of R_x (again the width of R_x is chosen long enough so that this can be done). We also place Γ_2 as the same manner as before to complete the drawing; see Fig. 8(c).

Finally if $x < i$, then we draw $G[1, x]$ by the **Staircase Algorithm** and delete from it 1 and x to obtain Γ_1 . However, to compute Γ_2 , we recursively draw $G[x, n]$ and delete x and n from it. We now draw R_1, R_x and R_n as before but this time the right side of R_n should extend past R_x . We now place Γ_1 (after possible scaling) touching the right side of R_1 and the top side of R_x (this is again possible for suitable choice of the height of R_1). Finally we complete the drawing by placing Γ_2 (after possible scaling) touching the right side of R_x and the top side of R_n so that the right side of the rectangle for $n - 1$ extends past R_n ; see Fig. 8(d). \square

Note that in the layout obtained above the top-right corners for vertices $\{1, \dots, i\}$ and the bottom-right corners for vertices $\{i + 1, \dots, n\}$ form two staircases. Thus we refer to this as a **Double-Staircase** layout, to the algorithm as the **Double-Staircase Algorithm**, and to vertex i as the *pivot vertex*.

Let O be a maximal outerplanar graph and let Γ be either a **Staircase** or a **Double-Staircase** layout. Then any triangle $\{p, q, r\}$ in O is represented by three rectangles and the shared boundaries of these rectangles define a *T-shape*. The vertex whose two shared boundaries are collinear in the T-shape is called the *pole* of the triangle $\{p, q, r\}$.

Proof of Theorem 4. Let \mathcal{T} be the construction tree for G . We compute a representation for G by a top-down traversal of \mathcal{T} , constructing the representation for each piece as we traverse it. Let P be a piece of G at the l -th level. If P is the root of \mathcal{T} , then we use the **Staircase Algorithm** to find a contact representation of P with rectangles in the plane and then we give necessary heights to these rectangles to obtain a proportional contact representation of P with boxes. Otherwise, the vertices of P are adjacent to exactly three $(l - 1)$ -level vertices A, B, C that form a triangle in the parent piece of P . Since A, B, C belong to the parent piece of P , their boxes have already been drawn when we start to draw P . To find a correct representation of G , we need that the boxes for

the vertices in P have correct adjacencies with the boxes for A , B , and C ; hence we assume a fixed structure for such a triangle. We maintain the following invariant:

Let $\{p, q, r\}$ be three vertices in a piece P of G forming a triangle. Then in the proportional contact representation of P , the boxes for p, q, r are drawn in such a way that (i) the projection of the mutually shared boundaries for these boxes in the xy -plane forms a T-shape, (ii) the highest faces (faces with largest z -coordinate) of the three rectangles have different z coordinates and the highest face of the pole-vertex of the triangle has the smallest z -coordinate.

Note that by choosing the areas of the rectangles in the **Staircase** layout, we can maintain this invariant for the parent piece by appropriately adjusting the heights of the boxes (e.g., incrementally increasing heights for the vertices in the recursive **Staircase Algorithm**).

We now describe the construction of a proportional box-contact representation of P with the correct adjacencies for A , B and C . By the invariant the projection of the shared boundaries for $\{A, B, C\}$ forms a T-shape in the xy -plane. Without loss of generality assume that A is the pole of the triangle and the highest faces of B , C and A are in this order according to decreasing z -coordinates. Also assume that P is a maximal outerplanar graph; we later argue that this assumption is not necessary.

Let ab be a common neighbor of A and B ; bc a common neighbor of B and C ; ca a common neighbor of C and A . Then the outer cycle of P can be partitioned into three paths: P_a is the path from ca to ab , P_b is the path from ab to bc and P_c is the path from bc to ca . All the vertices on the path P_a (P_b , P_c , respectively) are adjacent to A (B , C , respectively). We first assume that there is no chord in P between ca and a vertex on path P_a . We consider the following two cases.

Case 1: No vertex of P is adjacent to all of $\{A, B, C\}$. We label the vertices of P in the clockwise order starting from $ca = 1$ and ending at n , where n is the number of vertices in P . Let i and j be the indices of vertices bc and ab , respectively. Let x be the index of the vertex that is the (unique) third vertex of the inner face of P containing the edge $(1, n)$. Define the two graphs $G[1, x]$ and $G[x, n]$ as in the proof of Lemma 4. We first find a proportional contact representation of P for w restricted to the vertices of P using rectangles in the plane, then we give necessary heights to this rectangles. Draw $G[x, n]$ using the **Staircase Algorithm** and delete the rectangles for x and n to obtain Γ_2 . Draw rectangles R_x and R_n for x and n , respectively, so that the bottom side of R_x touches the top side of R_n , the left sides for both the rectangles have the same x -coordinate and the right side of R_n extends past R_x . Now place Γ_2 (after possible scaling) touching the right side of R_x and the top side of R_n (this is possible since we can make the width of R_n sufficiently long); see Fig. 9. Place the rectangle R_1 for 1 touching the left sides of R_x and R_n such that its bottom side is aligned with R_n and its top side is aligned with the top side of the rectangle for j . To complete the rest of the drawing, we have the following two subcases:

Case 1a: $x \leq i$. We draw $G[1, x]$ using the **Staircase Algorithm** and delete from it the rectangles for 1 and x to obtain Γ_1 . We finally place Γ_1 (after 90° counterclockwise rotation and possible scaling) touching the top side of R_1 and left side of R_x (this is possible by choosing a sufficiently large height for R_x); see Fig. 9(a).

Case 1b: $x > i$. We draw $G[1, x]$ using the **Double-Staircase Algorithm** where i is the pivot vertex. From this drawing, we delete the rectangles for 1 and x to obtain I_1 . Finally place I_1 (after 90° counterclockwise rotation and possible scaling) touching the top side of R_1 and left side of R_x such that the topside of the rectangle for $(x - 1)$ goes past the top side of R_x ; see Fig. 9(b).

So far we used the function w to assign areas for the rectangles and obtained proportional box-contact representation of P from the rectangles by assigning unit heights. However, by changing the areas for the rectangles, we can obtain different heights for the boxes. We will use this property to maintain adjacencies with $\{A, B, C\}$, as well as to maintain the invariant. Specifically, once we get the box representation of P , we scale it by increasing the heights for the boxes, so that when we place it at the corner created by the T-shape for $\{A, B, C\}$ it will not intersect the representation for any of its sibling pieces in \mathcal{T} . Consider the point p which is the intersection of the lines containing the right side of the rectangle for i and the top side of the rectangle for j . We place I such that the point p superimposes on the corner for the T-shape in the projection on the xy -plane. Since the highest faces of B, C and A are in this order according to z -coordinate, the adjacencies of the vertices in P with $\{A, B, C\}$ are correct. By appropriately choosing the areas for the rectangles, we ensure that all the boxes for the vertices of P have their highest faces above that of B and that the invariant is maintained.

Case 2: A vertex of P is adjacent to all of $\{A, B, C\}$. In this case at least one of $\{A, B, C\}$ has only one neighbor in P . Assume first that a vertex b ($= ab = bc$) of P is adjacent to all of $\{A, B, C\}$ and this is the only neighbor of B ; see Fig. 9(c). Then we follow the steps for *Case 1a* with $b = j$ (and some vertex between x and b as i). But when we finally place this representation of P on the corner for the T-shape of $\{A, B, C\}$ we find the point p to superimpose on this corner as follows. The point p is on the line containing the top side of the rectangle for b and has x -coordinate between the right sides of the rectangles for b and $(b - 1)$.

If a vertex c ($= bc = ca$) is adjacent to all of $\{A, B, C\}$ and is the only neighbor of C in P , then we follow the steps of *Case 1b* with $i = 2$ and $j = ab$; see Fig. 9(d). We find the point p to superimpose on the corner for the T-shape of $\{A, B, C\}$ as follows. The point p is on the line containing the top side of the rectangle for $j = ab$ and has x -coordinate between the left sides of the rectangles for 1 and 2.

If a vertex a ($= ab = ca$) is adjacent to all of $\{A, B, C\}$ and is the only neighbor of A in P , then we number the vertices of P in the clockwise order starting from the clockwise neighbor of a and ending at $a = n$; see Fig. 9(e). We use the **Staircase Algorithm** to find a representation of P with rectangles and give necessary heights to obtain a representation with boxes. On the corner for the T-shape of $\{A, B, C\}$, we superimpose the intersection point for the lines containing the top side of the rectangle of n and the right side of the rectangle for bc .

Finally, we consider the case when there is a chord between ca and another vertex on the path P_a . Take the innermost such chord and let its other end-vertex be t . Then consider the two subgraphs P_1 and P_2 induced by all the vertices outside the chord and inside the chord (along with the two vertices ca and t). P_1 does not contain any chord from ca ; thus we use the algorithm above to obtain a representation of P_1 ; denote this by I' . In this representation ca and t will play the roles of 1 and n , respectively. Each

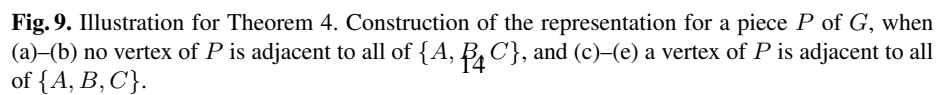


Fig. 9. Illustration for Theorem 4. Construction of the representation for a piece P of G , when (a)–(b) no vertex of P is adjacent to all of $\{A, B, C\}$, and (c)–(e) a vertex of P is adjacent to all of $\{A, B, C\}$.

vertex of P_2 is adjacent to A and we find a proportional contact representation of P_2 and attach it with Γ' as follows. We use the **Staircase Algorithm** to find a proportional contact representation of P_2 with rectangles in the plane and delete the rectangles for ca and t from it to obtain Γ'' . In Γ' , we change the height of the rectangle R_1 for $ca = 1$ to increase its area so that its bottom side extends past the bottom side of the rectangle R_n for $t = n$. Then we place Γ'' (after reflecting with respect to the x -axis and possible scaling) touching the right side of R_1 and the bottom side of R_n . Since the **Staircase Algorithm** can accommodate any given area for the layout, we can change the heights of the boxes for the vertices in P_2 to maintain the invariant.

Thus with the top-down traversal of \mathcal{T} , we obtain a proportional contact representation for O . We assumed that each piece of O is maximal outerplanar. However in the contact representation, for each edge (u, v) , either a face of the box R_u for u is adjacent to the box R_v for v and no other box; or a face of R_v is adjacent to R_u and no other box. In both cases the adjacency between these two boxes can be removed without affecting any other adjacency. Thus this algorithm holds for any nested outerplanar graph O . \square

6 Conclusions and Future Work

We proved that nested maximal outerplanar graphs have cube-contact representations and nested outerplanar graphs have proportional box-contact representations. These classes of graphs are special cases of k -outerplanar graphs, and the set of k -outerplanar graphs for all $k > 0$ is equivalent to the class of all planar graphs. Even though our approach might generalize to large classes, cube-contact representations and proportional box-contact representations are still open for general planar graphs.

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